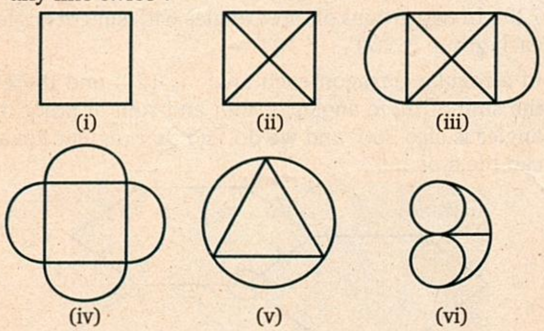
Olympiad Track

10 Questions from previous Maths Olympiads discussed

1. Which of the following figures you can trace without lifting your pencil from the paper and without tracing any line twice ?



In each case find the number of line segments starting from every point or vertex. Do you find any relation between the number of these lines and the possibility of tracing the figure without lifting the pencil ? Can you give a logical reason for the result you have obtained ?

Sol. It will be found that figures (i), (iii), (iv), (v), can be traced without lifting the pencil and tracing any line twice, but figures (ii) and (vi) cannot be so traced.

It will also be found that in figures (ii), (vi), all vertices are of odd degree while in figures (i), (iii), (iv), (v), all the vertices are of even degree. (*i.e.*, in figures (ii) and (vi) odd number edges meet at every vertex while in figures (i), (iii), (iv), (v) even number of edges meet at every vertex).

Thus, if at a vertex, an even number of edges meet, we can start as well as end at that vertex, but if at a vertex, odd number of edges meet, we cannot start our path at a vertex and also end the path there.

2. If

$$\begin{array}{cccc} a & b-y & c+y & d \\ c-x & d & a & b+x \\ d+x & c & b & a-x \\ b & a+y & d-y & c \end{array}$$

is magic square with magic constant $a+b+c+d$ provided $x < a, c$ and $y < b,\ d$, how many magic squares can you get with $22-12-18-87$ as elements of the first row ?

Sol. Let $y = 1$, then $a = 22,\ b = 13,\ c = 17$ and $d = 87$ and the magic square is

$$\begin{array}{cccc} 22 & 12 & 18 & 87 \\ 17-x & 87 & 22 & 13+x \\ 87+x & 17 & 13 & 22-x \\ 13 & 23 & 86 & 17 \end{array}$$

$\Rightarrow \quad x = 1,\ 2, \dots, 16$ ... (1)

Let $y = 2$, then $a = 22,\ b = 14,\ c = 16,\ d = 87$.

So that the magic square is

$$\begin{array}{cccc} 22 & 12 & 18 & 87 \\ 16-x & 87 & 22 & 14+x \\ 87+x & 16 & 14 & 22-x \\ 14 & 24 & 85 & 16 \end{array}$$

$\Rightarrow \quad x = 1,\ 2,\ 3, \dots, 16$ ... (2)

But, putting $y = 1,\ 2,\ 3, \dots, 16$, we get total number of magic squares

$= 16 + 15 + \dots + 1$

$= 136$

with $22-12-18-87$ in the first row.

3. Can you form a magic square where the 4 corner elements or the 4 central elements or the 4 elements of one of the four corner 2×2 squares give your date of birth ? If so, form these squares.

Sol. In the magic latin square we know each of a, b, c, d, if we know any one of the following

$$\begin{array}{cccc} \underline{\underline{a}} & b & c & \underline{\underline{d}} \\ \underline{\underline{c}} & d & a & \underline{\underline{b}} \\ d & \underline{\underline{c}} & \underline{\underline{b}} & a \\ \underline{\underline{b}} & \underline{\underline{a}} & \underline{\underline{d}} & \underline{\underline{c}} \end{array}$$

(i) The four corner elements $\begin{matrix} a & d \\ b & c \end{matrix}$

(ii) The four central elements $\begin{matrix} d & a \\ c & b \end{matrix}$

(iii) The elements of any of the four 2×2 corner squares *i.e.*,

$\begin{matrix} ab \\ cd \end{matrix} \qquad \begin{matrix} cd \\ ab \end{matrix} \qquad \begin{matrix} dc \\ ab \end{matrix} \qquad \begin{matrix} ba \\ dc \end{matrix}$

(iv) Four elements of any of the four rows.

(v) Four elements of any of the four columns.

(vi) Four elements of either of two diagonals.

(vii) Four underlined elements or four double underlined elements in each of these cases we know a, b, c, d and knowing these we can find the magic square.

Let the date of birth be 02 – 12 – 1978.

Thus, if the four central elements give (02–12–1978) as,

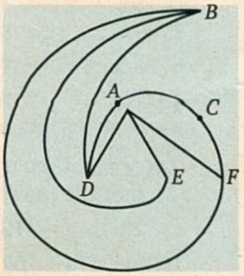
$d = 02, \quad a = 12, \quad c = 19, \quad b = 78$

So, the corresponding magic square is

12	78	19	02
19	02	12	78
02	19	78	12
78	12	02	19

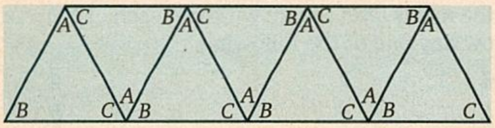
4. Each of the three homes A, B and C has to be connected to each of the three service points D, E, F by non-intersecting wires in the plane. Can you do it ? If not, what is the reason ?

Sol. We shall find that we can join two of points A, B, C to each of D, E, F by non-intersecting line and we can also joint the third of A, B, C to two of D, E, F but we cannot join third to each of the three points D, E, F.If we could do that, we would have a graph with six vertices each of odd degree 3 and edges 3 with all edges not-intersecting it appears that is not possible.

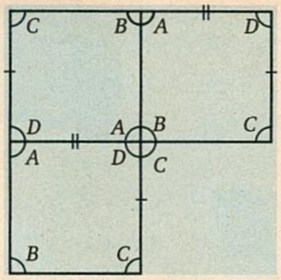


5. Make 16 identical triangles of any shape. Try to cover the top of your table so that no gaps are left. Are your always successful and if so why ? Similarly, make 16 identical quadrilateral of any shape. Again try to cover the top of your table with these quadrilateral without leaving any gaps. Do you always succeed and if so why ?

Sol. We always succeed with identical triangles of any shape. The reason for the success is clear from the figure and arises due to the sum of the angles of every triangle being 180° and this also being the sum of angles at a point on one side of a straight line.



The reason for the success for a quadrilateral is that the sum of angles of a quadrilateral is 360° and the sum of angles at a point is also 360°.

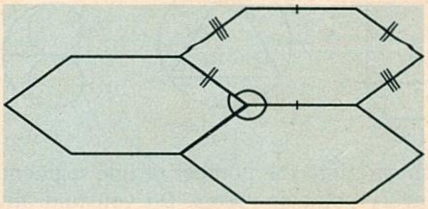


It is important to note that simple geometrical theorems about the sum of angles of a triangle or quadrilateral or sum of angles at a point can have such a significant influence for covering the plane with identical triangles and quadrilaterals.

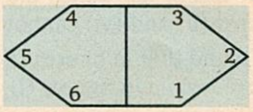
6. Repeat the above experiment with identical pentagons and hexagons of any shape. Do you always succeed or do you succeed sometimes ? Try to explain the reasons for the same. Repeat the above experiment with regular hexagons and hexagons with equal and opposite parallel sides, but not all sides equal. Why do you succeed in these 2 cases ?

Sol. We don't succeed with identical pentagons or hexagons of arbitrary shapes. The sum of angles of a convex pentagon is 540°, similarly for hexagon sum of angles is 720°. In two groups of three angles with sum of angles of each group is 360°.

In a regular hexagon each angle is 120° and therefore the sum of three angles is 360° and sum of other three angles is also 360° and we do find that regular hexagon can tile a plane.



If we take a hexagon with pairs of equal and opposite sides (but all six sides need not be equal), the opposite angles will be equal and the sum of any three consecutive angles will be 360°. Such a hexagon can also tile a plane.



If we join the middle points of two opposite sides of a hexagon of this type (which includes regular hexagons as a special case), we get two congruent pentagons in each of which the sum of three angles is 360° and the sum of two angles is 180° and each can tile the plane.

7. Can you cover the top of the table with identical circles or even with circles of different radii without leaving any gaps ? Can you succeed with ellipses or parabolas or any other figures whose boundaries are not straight lines ?

Sol. It is obvious that identical circles alone cannot tile a plane, but identical circles of radius r together with identical regions bounded by quarter circular arcs of radius r can easily do this.

Similarly, identical ellipses cannot tile a plane, but identical ellipses together with identical regions bounded by 4 quarter ellipses can do this.

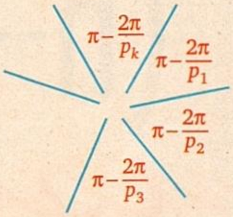
Parabolas are completely unsuitable for tiling a plane.

However, tiling by regions bounded by sine curves or cosine curves or by circular curves or other periodic curves could be successful.

Also, tiling by modifications of squares and hexagonal tiles are also possible.

8. Suppose regular polygons of number of sides $p_1, p_2, \ldots, p_k$ meet a common vertex, so that no gap is left. What is the relation between $p_1, p_2, \ldots, p_k$.

Sol. The sum of the exterior angles of a convex polygon of any number of sides is 2π. For a regulars polygon of p sides, each exterior angle would be $2\pi / p$ and each interior angle would be $\pi - \dfrac{2\pi}{p}$. If k polygons of sides



$p_1, p_2, \ldots, p_k$ meet at a point and there is no gap, then the sum of the angles at a point is 2p, it follows that

$$\left(\pi - \frac{2\pi}{p_1}\right) + \left(\pi - \frac{2\pi}{p_2}\right) + \ldots + \left(\pi - \frac{2\pi}{p_k}\right) = 2\pi.$$

or $$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} = \frac{k}{2} - 1.$$

9. Find all possible solutions of the equation obtained in the previous problem in positive integers, each ≥ 3.

Sol. Case 1 : $k = 6$

Our equation becomes

$$\left(\frac{1}{3} - \frac{1}{p_1}\right) + \left(\frac{1}{3} - \frac{1}{p_2}\right) + \ldots + \left(\frac{1}{3} - \frac{1}{p_6}\right) = 0$$

of which the only solution is

$p_1 = p_2 = \ldots = p_6 = 3$

which we denote by (3, 3, 3, 3, 3, 3) which means that 6 triangles meet at every vertex.

Case 2 : $k = 5$

In this case, we get

$$\left(\frac{1}{3} - \frac{1}{p_1}\right) + \left(\frac{1}{3} - \frac{1}{p_2}\right) + \ldots + \left(\frac{1}{3} - \frac{1}{p_5}\right) = \frac{1}{6}$$

which has 2 solutions, namely (3, 3, 3, 3, 6) and (3, 3, 3, 4, 4)

In the first case 4 triangles and 1 hexagon meet at each vertex and in the second case, 3 triangles and 2 squares meet at each vertex.

Case 3 : $k = 3$.

In this case, we get

$$\left(\frac{1}{3} - \frac{1}{p_1}\right) + \left(\frac{1}{3} - \frac{1}{p_2}\right) + \left(\frac{1}{3} - \frac{1}{p_3}\right) + \left(\frac{1}{3} - \frac{1}{p_4}\right) = \frac{1}{3}.$$

This has 4 solutions, namely (3, 3, 4, 12), (3, 3, 6, 6), (3, 4, 4, 6), (4, 4, 4, 4). In the first case 2 triangles, 1 square and 1 dodecagon meet at each vertex. In the second case 2 triangles and 2 hexagons meet at each vertex. In the third case 1 triangle, 2 squares and 1 hexagon meet at each vertex. In the fourth case 4 squares meet at each vertex.

Case 4 : $k = 3$.

Here we get

$$\left(\frac{1}{3} - \frac{1}{p_1}\right) + \left(\frac{1}{3} - \frac{1}{p_2}\right) + \left(\frac{1}{3} - \frac{1}{p_3}\right) = \frac{1}{12}$$

This gives the solution as,

(3, 7, 42), (3, 8, 24), (3, 9, 6), (3, 12, 12), (4, 5, 30), (4, 6, 12), (4, 8, 8), (6, 6, 6).

Thus, we get 15 solutions in all. However, it is not necessary that each of these gives a tessellation of the plane nor it implies that each solution may give rise to only one tessellation.

10. Join the point (1, 0) to (0, 10); (2, 0) to (0, 9); (3, 0) to (0, 8) and so on till you join (10, 0) to (0, 1) by straight lines segments. Do all these line segments appear to touch a curve ? What is this curve ?

Sol. The equation of every line is of the form

$$\frac{x}{a} + \frac{y}{11-a} = 1$$

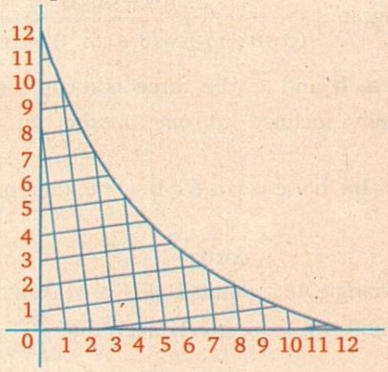
for different values of a, it will give a family of straight lines.

If the line of this family passes through a given point (x, y) we get

$$a(11-a) - ay - (11-a)x = 0$$

or $$a^2 + a(y - x - 11) + 11x = 0$$

Thus, in general, two lines of the family will pass through a given point.

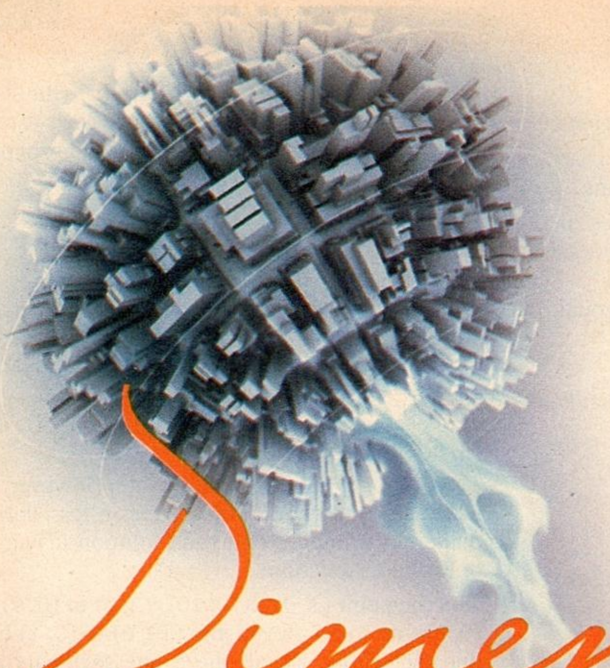


However, if

$$(y - x - 11)^2 - 44x = 0$$

or $$(y - x)^2 - 22y - 22x + 121 = 0,$$

the two lines will coincide so that the two coincident lines will give a tangent to the curve represented by this curve so that the various lines will touch the curve which is a parabola, since the terms of second degree forms a perfect square. The parabola, which touches every member of a family of straight lines, is called the *envelope of the family of straight lines.*



The $(n+1)^{th}$ Dimension

These problems, in this format, are not asked in the JEE these days, but they give you invaluable insights into tackling the latest JEE pattern

1. **(a) Find the domain and range of**
$$f(x) = \ln \frac{1}{\sqrt{[\cos x] - [\sin x]}},$$
where [ ] denotes the greatest integer function.
(b) Solve the equation
$$\left| |x^2 - 5x + 4| - |2x - 3| \right| = |x^2 - 3x + 1|.$$

2. Integrate $\displaystyle\int \frac{2\sin\theta + \sin 2\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}\, d\theta$.

3. Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ be three non-coplanar unit vectors equally inclined to one another at an angle θ.
If $\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} = p\,\vec{\mathbf{a}} + q\,\vec{\mathbf{b}} + r\,\vec{\mathbf{c}}$, then prove that
$$p^2 + \frac{q^2}{\cos\theta} + r^2 = 2.$$

4. In triangle ABC, points D, E, F are taken on the sides BC, CA and AB, respectively such that $\dfrac{BD}{DC} = \dfrac{CE}{EA} = \dfrac{AF}{FB} = n$.
Prove that $(\text{ar}\,\Delta\, DEF) = \dfrac{n^2 - n + 1}{(n+1)^2}(\text{ar}\,\Delta\, ABC)$.

5. In the scalene triangle ABC, the altitudes AP and CQ are dropped from the vertices A and C to the sides BC and AB. The area of the triangle ABC is equal to 18, the area of $\Delta\, BPQ$ is 2 and length of the segment PQ is $2\sqrt{2}$. Prove that distance between centroid of $\Delta\, ABC$ and vertex B is $2\sqrt{2 + \sqrt{2}}$.

6. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \le 1$$
when does the equality hold ?

7. Let there be terms in GP $a_1, a_2, a_3, \ldots, a_n$ whose common ratio is r. Let S_k denotes the sum of first k terms of this GP. Prove that
$$S_{m-1}S_m = \frac{r+1}{1}\sum_{i<j}^{m} a_i a_j .$$

8. Find all the values of the parameter $a\,(a \ge 1)$ for which the area of the figure bounded by the pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]\,x^2$, $y = \dfrac{1}{2}[a]x^2$ is the greatest, where [ ] denotes the greatest integer function.

9. $y^2 = 2ax$ is a parabola on which $A_1(t_1)$, $A_2(t_2)$, ..., $A_{n-2}(t_{n-2})$, $A_{n-1}(t_{n-1})$, $A_n(t_n)$ are the points in order. Out of these, except A_{n-1}, all are points. A polygon is formed by joining these points. Find the coordinates of A_{n-1} on the parabola so that the area of this polygon is the greatest.

10. From $4m+1$ tickets numbered as 1, 2, $4m+1$, three tickets are chosen at random. Find out the probability that the numbers are in AP, with even common difference.

11. Solve the differential equation :
$$(x\,dy + y\,dx)\sin(xy) + (x^2y\,dy + y^2x\,dx)\cos(xy) = 0.$$

12. Consider a set $A = \{1, 2, 3, \ldots\ldots, n\}$. Let B_i be any subset of A consisting of three elements and a_i be the least element of B_i. Find the arithmetic mean of a_is.

13. Consider an equilateral triangle ABC. Vertices B and C lie on the line $x = y = 2$. Find the coordinates of vertices A, B, C given that centroid of the triangle is $(\sqrt{2},\ 2)$.

14. Consider two circles; $S_1 : x^2 + y^2 - 1 = 0$ and $S_2 : x^2 + y^2 - 9 = 0$. A parabola is made to pass through the points of intersection of S_1 with x-axis and to have

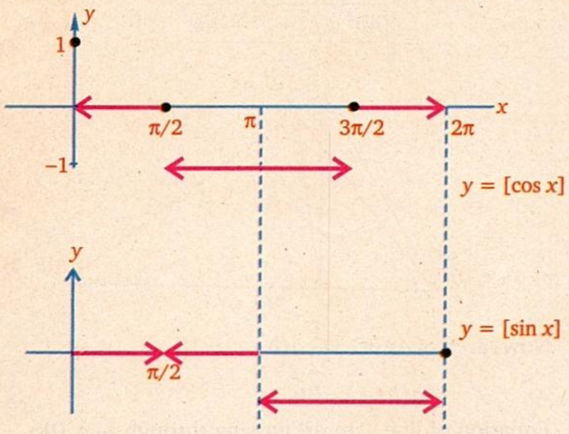
any tangent of S_2 as its directrix. Find the locus of the focus of all such parabolas.

15. Let G be the centroid of triangle ABC, where $\angle ABC$, where $\angle C > \frac{\pi}{2}$. AD and CF are the medians from angular points A and C. If the points B, D, G and F are concyclic, prove that

$$\frac{BC}{AC} < \frac{1}{\sqrt{2}}.$$

EXPLANATIONS

1. (a) $f(x)$ is defined when $[\cos x]-[\sin x]>0$



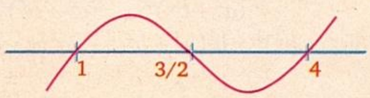
From above diagram,

$$[\cos x]-[\sin x]>0$$

$\Rightarrow$ Domain, $x=\{0\}\cup\left[\frac{3\pi}{2},\ 2\pi\right]$ where $n=0,\ \pm 1,\ldots.$

Range of function $=\{\log\ 1\}=\{0\}$.

(b) Given equation can be written as



$$\left||x^2-5x+4|-|2x-3|\right|=|x^2-5x+4+2x-3|$$

On squaring both sides (as both sides are +ve)

$$(x^2-5x+4)^2+(2x-3)^2-2|x^2-5x+4||2x-3|$$
$$=(x^2-5x+4)^2+(2x-3)^2+2(x^2-5x+4)(2x-3)$$
$$\Rightarrow \qquad -2|x^2-5x+4|\,|2x-3|$$
$$=2(x^2-4x+4)(2x-3)$$
$$\Rightarrow |(x^2-5x+4)(2x-3)|=-(x^2-5x+4)(2x-3)$$
$$\Rightarrow \qquad (x^2-5x+4)(2x-3)\le 0$$
$$\Rightarrow \qquad (x-1)(x-4)(x-3/2)\le 0$$

By *Wavy curve method*

$$x\in(-\infty,\ 1]\cup\left[\frac{3}{2},\ 4\right].$$

2. Let

$$I=\int\frac{2\sin\theta+\sin 2\theta}{(\cos\theta-1)\sqrt{\cos\theta+\cos^2\theta+\cos^3\theta}}\,d\theta$$

Put $\cot\theta=x^2-\sin\theta\ d\theta=2x\ dx$.

$$I=2\int\frac{1+x^2}{1-x^2}\cdot\frac{2x\,dx}{\sqrt{x^2+x^4+x^6}}$$

$$=4\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(\frac{1}{x}-x\right)\sqrt{\left(\frac{1}{x}-x\right)^2+3}}$$

Put $\frac{1}{x}-x=t\Rightarrow -\left(\frac{1}{x^2}+1\right)dx=dt$

$$I=4\int\frac{-dt}{t\sqrt{t^2+3}}$$

Again put $t^2+3=u^2$

$$2t\ dt=2u\ du\Rightarrow t\ dt=u\ du$$

$$I=4\int\frac{-u\,du}{u(u^2-3)}=-4\int\frac{du}{u^2-3}$$

$$=-\frac{2}{\sqrt{3}}\ln\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right|+c=-\frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right|$$

$$=-\frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right|+C$$

$$=-\frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{(\cos\theta+\sec\theta+1)}-\sqrt{3}}{\sqrt{(\cos\theta+\sec\theta+1)}+\sqrt{3}}\right|+C.$$

3. $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}=\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}=\vec{\mathbf{c}}\cdot\vec{\mathbf{c}}=1$

and $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}=\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}=\cos\theta$

Consider, $\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{b}}\times\vec{\mathbf{c}}=p\vec{\mathbf{a}}+q\vec{\mathbf{b}}+r\vec{\mathbf{c}}$

Taking dot product with $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, respectively, we get

$$[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]=p+(q+r)\cos\theta \qquad \ldots(1)$$
$$0=q+(p+r)\cos\theta \qquad \ldots(2)$$
$$[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]=\cos\theta\,(p+q)+r \qquad \ldots(3)$$

On adding Eqs. (1), (2) and (3), we get

$$2[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]=(p+q+r)+2(p+q+r)\cos\theta$$
$$2[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]=(p+q+r)(1+2\cos\theta) \qquad \ldots(4)$$

On solving Eqs. (1) and (4), we get

$$p=\frac{[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]}{(1+2\cos\theta)(1-\cos\theta)} \qquad \ldots(5)$$

On solving Eqs. (2) and (4), we get

$$q=-\frac{2[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]\cos\theta}{(1+2\cos\theta)(1-\cos\theta)} \qquad \ldots(6)$$

On solving Eqs. (3) and (4), we get

$$r=\frac{[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]}{(1+2\cos\theta)(1-\cos\theta)} \qquad \ldots(7)$$

Now to find $[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]$

$$[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]^2 = \begin{vmatrix} \vec{\mathbf{a}}.\vec{\mathbf{a}} & \vec{\mathbf{a}}.\vec{\mathbf{b}} & \vec{\mathbf{a}}.\vec{\mathbf{c}} \\ \vec{\mathbf{b}}.\vec{\mathbf{a}} & \vec{\mathbf{b}}.\vec{\mathbf{b}} & \vec{\mathbf{b}}.\vec{\mathbf{c}} \\ \vec{\mathbf{c}}.\vec{\mathbf{b}} & \vec{\mathbf{c}}.\vec{\mathbf{b}} & \vec{\mathbf{c}}.\vec{\mathbf{c}} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$,

$$[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]^2 = \begin{vmatrix} 1+2\cos\theta & 1+2\cos\theta & 1+2\cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & 1 & 1 \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1+2\cos\theta)[(1-\cos^2\theta) + 1(\cos^2\theta - \cos\theta) + 1(\cos^2\theta - \cos\theta)]$$

$$= (1+2\cos\theta)[1+\cos^2\theta - 2\cos\theta]$$

$$= (1-\cos\theta)^2(1+2\cos\theta)$$

$$\Rightarrow [\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}] = (1-\cos 2\theta)\sqrt{1+2\cos\theta}$$

$$\text{Now, } p^2 + \frac{q^2}{\cos\theta} + r^2 = \frac{[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}]^2[2+4\cos\theta]}{(1+2\cos\theta)^2(1-\cos\theta)^2}$$

$$= \frac{(1-\cos\theta)^2(1+2\cos\theta)2(1+2\cos\theta)}{(1+2\cos\theta)^2(1-\cos\theta)^2}$$

$$= 2 = \text{RHS}$$

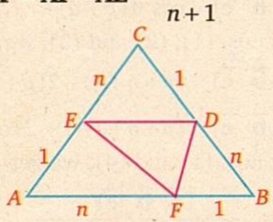
4. Take A as the origin and let the position vectors of the point B and C be $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ respectively.

Therefore, the position vectors of D, E, F are respectively,

$$\frac{n\vec{\mathbf{c}}+\vec{\mathbf{b}}}{n+1}, \frac{\vec{\mathbf{c}}}{n+1}, \frac{n\vec{\mathbf{b}}}{n+1}$$

$$\therefore \quad \vec{FD} = \vec{AD} - \vec{AF} = \frac{n\vec{\mathbf{c}}+\vec{\mathbf{b}} - n\vec{\mathbf{b}}}{n+1}$$

$$\text{and } \vec{EF} = \vec{AF} - \vec{AE} = \frac{n\vec{\mathbf{b}} - \vec{\mathbf{c}}}{n+1}$$



Now, the vector area of $\Delta\, ABC = \frac{1}{2}(\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ and the vector area of $\Delta\, DEF = \frac{1}{2}(\vec{FD} \times \vec{FE})$

$$= \frac{1}{2(n+1)^2}[(n\vec{\mathbf{b}} - \vec{\mathbf{c}})\{(n\vec{\mathbf{c}} + (1-n)\vec{\mathbf{b}})\}]$$

$$= \frac{1}{2(n+1)^2}[n^2\vec{\mathbf{b}} \times \vec{\mathbf{c}} + (1-n)\vec{\mathbf{b}} \times \vec{\mathbf{c}}]$$

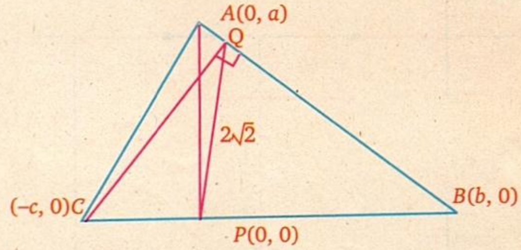
$$= \frac{1}{2(n+1)^2}[(n^2 - m + 1)(\vec{\mathbf{b}} \times \vec{\mathbf{c}})]$$

$$= \frac{n^2 - n + 1}{(n+1)^2}\Delta\, ABC.$$

5. Centroid of $\Delta\, ABC$, $G = \left(\frac{b-c}{3}, \frac{a}{3}\right)$

$$\text{Distance } GB = \sqrt{\left(\frac{b-c}{3} - b\right)^2 + \left(\frac{a}{3}\right)^2}$$

$$= \sqrt{\frac{4b^2 + c^2 + a^2 + 4bc}{9}} \quad \ldots (1)$$



$$\text{Now, area of } \Delta ABC = \frac{1}{2}a(b+c) = 18$$

$$\Rightarrow \quad a(b+c) = 36 \quad \ldots (2)$$

Equation of line $\perp$ to AB passing through $(-c, 0)$,

$$bx - ay = -bc$$

Point of intersection of these lines is

$$Q \equiv \left(\frac{b(a^2 - bc)}{a^2 + b^2}, \frac{ab(b+c)}{a^2+b^2}\right) \quad \ldots (3)$$

Now, given that $PQ = 2\sqrt{2}$

$$\Rightarrow \quad PQ^2 = \frac{b^2(a^2 - bc)^2}{(a^2+b^2)^2} + \frac{a^2b^2(b+c)^2}{(a^2+b^2)^2} = (2\sqrt{2})^2.$$

$$\Rightarrow \quad 8(a^2+b^2)^2 = b^2[a^4 + b^2c^2 - 2a^2bc + a^2b^2 + a^2c^2 + 2a^2bc]$$

$$= b^2(a^2+c^2)(a^2+b^2)$$

$$\Rightarrow \quad 8(a^2+b^2) = (a^2+c^2)b^2 \quad \ldots (4)$$

Now, we are given that area of $\Delta\, PBQ = 2$ sq. units

$$\Rightarrow \quad \frac{1}{2}b\frac{ab(b+c)}{(a^2+b^2)} = 2 \Rightarrow \frac{ab^2(b+c)}{(a^2+b^2)} = 4 \quad \ldots (5)$$

From Eqs. (4) and (5),

$$(a^2+c^2)b^2 = 2ab^2(b+c) \quad \ldots (6)$$

$$\Rightarrow \quad 2a(b+c) = a^2 + c^2$$

Also from Eqs. (2) and (5),

$$a^2 + b^2 = 9b^2 \Rightarrow a^2 = 8b^2 \quad \ldots (7)$$

$$4(a^2+c^2) = 36b^2$$

$$a^2 + b^2 = 9b^2 \Rightarrow a^2 = 8b^2 \quad \ldots (8)$$

$$\text{Also} \quad a^2 + b^2 = 2 \times 36 = 72 \quad \ldots (9)$$

$$\text{Now,} \quad GB = \frac{\sqrt{4b^2 + 4bc + c^2 + a^2}}{3}$$

We have, $a^2 + c^2 = 72$.

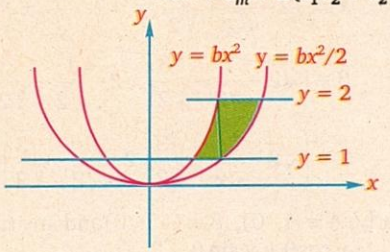
$$4b^2 + 4bc = 4b(b+c)$$

$$= 4b \times \frac{36}{a} = \frac{4b \times 36}{2\sqrt{2}b} = 36\sqrt{2}$$

$$\text{Hence,} \quad GB = \frac{\sqrt{72 + 36\sqrt{2}}}{3} = 2\sqrt{2+\sqrt{2}} \text{ units.}$$

6. We have, $(a^5+b^5)=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$
$=(a+b)\{(a-b)^2(a^2+ab+b^2)+a^2b^2\}\ge(a+b)a^2b^2$,
with equality if and only if $a=b$. Hence,
$$\frac{ab}{a^5+b^5+ab}\le\frac{ab}{a^2b^2(a+b)+ab}$$
$$\le\frac{c}{a+b+c}+\frac{a}{a+b+c}+\frac{b}{a+b+c}=1$$
and stated inequality is established.
Equality holds if $a=b=c=1$.

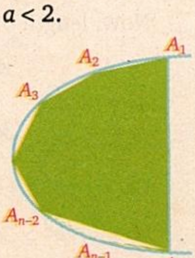
7. We have $(a_1+a_2+\ldots..+a_m)^2=a_1^2+a_2^2+\ldots\ldots$
$\ldots..+a_m^2+2(a_1a_2+a_2a_3+\ldots.)$



$$\text{or,}\quad\left[\frac{a_1(1-r^m)}{1-r}\right]^2=\frac{a_1^2(1-r^{2m})}{1-r^2}+2\sum_{i<j}^{m}a_ia_j$$
$$\Rightarrow\quad 2\sum_{i<j}^{m}a_ia_j=\frac{a_1^2(1-r^m)^2}{(1-r)^2}-\frac{a_1^2(1-r^{2m})}{1-r^2}$$
$$=\frac{2a_1^2}{(1-r)^2(1-r^2)}[r-r^m-r^{m+1}+r^{2m}]$$
$$=\frac{2r}{1+r}\left\{a_1\cdot\frac{(1-r^{m-1})}{1-r}\right\}\left\{\frac{a_1(1-r^m)}{1-r}\right\}$$
$$\text{i.e.,}\quad\frac{r+1}{1}\sum_{i<j}^{m}a_ia_j=S_{m-1}\cdot S_m.$$

8. Pair of lines; $y^2-3y+2=0\Rightarrow$ lines $y=2,\ y=1$, let $[a]=b$.
Now, curves are $y=bx^2$ and $y=\frac{b}{2}x^2$
$$\text{Bounded area}=2\left[\int_1^2(x_2-x_1)\,dy\right]$$
$$=2\left[\int_1^2(x_2-x_1)\,dy\right]=2\left[\int_1^2\left(\sqrt{\frac{2y}{b}}-\sqrt{\frac{y}{b}}\right)dy\right]$$
$$=2\left[\sqrt{\frac{2}{b}}\cdot\frac{y^{3/2}}{3/2}-\frac{1}{\sqrt{b}}\frac{y^{3/2}}{3/2}\right]_1^2$$
$$=\frac{4}{3\sqrt{b}}(\sqrt{2}-1)(2\sqrt{2}-1).$$
Area will be maximum when $b=[a]$ is least as
$a\ge1\Rightarrow[a]_{\text{least}}=1\Rightarrow1\le a<2.$

9. As $A_1, A_2, \ldots\ldots, A_{n-2}, A_n$ are fixed points. So, area of polygon formed by these points as shown by the shaded region is fixed.
Now, variable area of required polygon is only the triangle formed by points A_{n-2}, A_{n-1} and A_n.



Let $A_{n-1}\equiv(at^2,\ 2at)$
$$\text{Area of }\Delta=\frac{1}{2}\begin{vmatrix}at_{n-2}^2 & 2at_{n-2} & 1\\ at^2 & 2at & 2\\ at_n^2 & 2at_n & 1\end{vmatrix}=f(t)$$
For area to be maximum, $f'(t)$ should be zero
$$\Rightarrow\quad\begin{vmatrix}at_{n-2}^2 & 2at_{n-2} & 1\\ 2at & 2a & 0\\ at_n^2 & 2at_n & 1\end{vmatrix}=0$$
$$\Rightarrow\quad at_{n-2}^2(2a)+2at\,(2at_n-2at_{n-2})+at_n^2(-2a)=0$$
$$t=\frac{t_n^2-t_{n-2}^2}{2(t_n-t_{n-2})}=\frac{t_n+t_{n-2}}{2}$$
$$A_{n-1}(\alpha,\beta)=(at^2,\ 2at)=\left(a\left(\frac{t_n+t_{n-2}}{2}\right)^2,\ 2a\left(\frac{t_n+t_{n-2}}{2}\right)\right).$$

10. Let d be the common difference. If $d=2$, we can take 3 numbers as
$(1, 3, 5), (2, 4, 6), \ldots\ldots, (4m-3,\ 4m-1,\ 4m+1)$
In this case, we have $4m-3$ triplets. If $d=4$, we can take 3 numbers as
$(1, 5, 9), (2, 6, 10), \ldots., (4m-7,\ 4m-3,\ 4m+1)$
and in this case we have $(4m-7)$ triplets.
... ...
... ...
... ...
If $d=2m$, we have only one triplet
$(1,\ 2m+1,\ 4m+1)$
Total number of ways of selecting such triplets is
$4m-3+4m-7+\ldots.+1$
$$=\frac{m}{2}(2+(m-1)4-m(2m-1)).$$
Total number of ways of selecting 3 numbers out of $4m+1$ numbers is ${}^{4m+1}C_3$.
Required probability
$$=\frac{6m(2m-1)}{(4m+1)4m(4m-1)}=\frac{3}{2}\frac{(2m-1)}{(16m^2-1)}.$$

11. $(x\,dy+y\,dx)\sin(xy)+(x^2y\,dy+y^2x\,dx)\cos(xy)=0$
$\sin(xy)\,d(xy)+xy\cos(xy).\,d(xy)=0$
Put $xy=t$
$\Rightarrow\quad\sin t\,dt+t\cos t\,dt=0$
Integrating, $\int\sin t\ dt+\int t\cos t\,dt=0$
$-\cos t+t\sin t-\int\sin t\,dt=0$
$\Rightarrow\quad t\sin t=C\Rightarrow xy\sin(xy)=C.$

12. Total subsets having three elements $={}^nC_3$
Let 'i' be the least element of subset B_i. Clearly, $1\le i\le n-2$.
Now, number of subset having the least element as 'i' $={}^{n-i}C_2$.

$$\therefore \text{ Arithmetic mean of } a_i = \frac{\sum_{i=1}^{n-2} ({}^{n-i}C_2)}{{}^nC_3}$$

$$= \frac{\sum_{i=1}^{n-2} (n-2+1-i)\,{}^{n-(n-2+1-i)}C_2}{{}^nC_3}$$

$$= \frac{\sum_{i=1}^{n-2} (n-1-i)\,{}^{1+i}C_2}{{}^nC_3}$$

$$= \frac{1}{{}^nC_3}\left((n-1)\sum_{i=1}^{n-2} {}^{1+i}C_2 - \sum_{i=1}^{n-2} i\,{}^{1+i}C_2\right)$$

Now, $\sum_{i=1}^{n-2} {}^{1+i}C_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^{n-1}C_2$

$$= {}^3C_3 + {}^3C_2 + {}^4C_2 + \dots + {}^{n-1}C_2$$

$$= {}^4C_3 + {}^4C_2 + \dots + {}^{n-1}C_2 = {}^nC_3$$

and, $\sum_{i=0}^{n-2} i\,.\,{}^{1+i}C_2 = \sum_{i=1}^{n-2} \frac{i(1+i)i}{2} = \frac{1}{2}\sum_{i=1}^{n-2} i^3 = \frac{1}{2}\sum_{i=1}^{n-2} i^2$

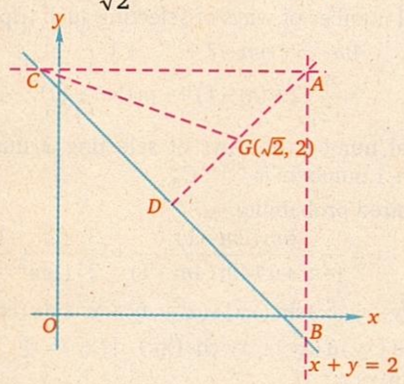
$$= \frac{n(n-1)(n-2)(3n-5)}{24} = {}^nC_3 \,.\, \frac{(3n-5)}{4}.$$

$$\Rightarrow \text{AM} = \frac{(n-1)\,.\,{}^nC_3 - \frac{1}{4}(3n-5)\,{}^nC_3}{{}^nC_3}$$

$$= \frac{1}{4}(4n-4-3n+5) = \frac{(n+1)}{4}.$$

13. Let the altitude drawn from centroid (G) meets the base at D.

$$\Rightarrow \quad DG = \frac{|\sqrt{2}+2-2|}{\sqrt{2}} = 1 \Rightarrow AG = 2.$$



Now, $CD = GD \cot 30°$

$\Rightarrow \quad BD = CD = \sqrt{3}.$

Slope of base is -1, hence slope of $GD = 1$.

Let $\quad A \equiv (x_A, y_A)$

$$\Rightarrow \quad \frac{x_A - \sqrt{2}}{1/\sqrt{2}} = \frac{y_A - 2}{1/\sqrt{2}} = 2$$

$$\Rightarrow \quad x_A = \sqrt{2} + \frac{2}{\sqrt{2}},\ y_A = 2 + \frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad A \equiv (2\sqrt{2},\ 2+\sqrt{2})$$

Similarly, for point D, we have

$$\frac{x_D - \sqrt{2}}{1/\sqrt{2}} = \frac{y_D - 2}{1/\sqrt{2}} = -1$$

$$\Rightarrow \quad x_D = \sqrt{2} - \frac{1}{\sqrt{2}},\ y_D = 2 - \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad D \equiv \left(\sqrt{2} - \frac{1}{\sqrt{2}},\ 2 - \frac{1}{\sqrt{2}}\right)$$

For points B and C, we get

$$\frac{x - x_D}{-1/\sqrt{2}} - \frac{y - y_D}{1/\sqrt{2}} = \pm\sqrt{3}$$

$$\Rightarrow \quad x = \sqrt{2} - \frac{1}{\sqrt{2}} \mp \sqrt{\frac{3}{2}},\ y = 2 - \frac{1}{\sqrt{2}} \pm \sqrt{\frac{3}{2}}$$

$$\Rightarrow \quad B \equiv \left(\sqrt{2} - \frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}},\ \sqrt{2} - \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\right)$$

and $\quad C \equiv \left(\sqrt{2} - \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}},\ 2 - \frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}}\right).$

14. Clearly, $A \equiv (1, 0)$, $B \equiv (-1, 0)$ and any tangent to S_2 is

$$x\cos\theta + y\sin\theta = 3 \qquad \dots (1)$$

Let the focus be $P(h, k)$. Now, by virtue of definition of parabola, distances of A and B from P should be same as the distances of A and B from eq. (1).

i.e., $\quad (h-1)^2 + k^2 = |\cos\theta - 3|^2 = (\cos\theta - 3)^2 \qquad \dots (2)$

Similarly,

$$(h+1)^2 + k^2 = (-\cos\theta - 3)^2 = (\cos\theta + 3)^2 \qquad \dots (3)$$

From Eq. (2) – Eq. (3), we get

$$(h-1)^2 - (h+1)^2 = (\cos\theta - 3)^2 - (\cos\theta + 3)^2$$

$$\Rightarrow \quad -4h = -6\cos\theta \Rightarrow \cos\theta = \frac{2h}{3}$$

On adding Eq. (2) and Eq. (3),

$$(h+1)^2 + k^2 + (h-1)^2 + k^2$$

$$= (\cos\theta - 3)^2 + (\cos\theta + 3)^2$$

$$\Rightarrow \quad h^2 + k^2 + 1 = \cos^2\theta + 9$$

$$\Rightarrow \quad h^2 + k^2 - 8 = \frac{4h^2}{9}$$

$$\Rightarrow \quad 5h^2 + 9k^2 - 72 = 0$$

Thus, locus is

$$5x^2 + 9y^2 - 72 = 0.$$

15. $\angle C > \frac{\pi}{2} \Rightarrow a^2 + b^2 - c^2 < 0$

$\Rightarrow \quad c^2 > a^2 + b^2$

Now, points $BDGF$ are concyclic

$\Rightarrow \quad AF\,.\,AB = AG\,.\,AD$

$$\Rightarrow \quad \frac{c}{2}\,.\,c = \frac{2}{3}AD\,.\,AD = \frac{2}{3}AD^2$$

$\Rightarrow \quad 3c^2 = 4AD^2$

Now, length of median through the angular point A,

$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$\Rightarrow \quad 3c^2 = 4\,.\,\frac{1}{4}(2b^2 + 2c^2 - a^2)$$

$$\Rightarrow \quad c^2 = 2b^2 - a^2 > a^2 + b^2$$

$$\Rightarrow \quad b^2 > 2a^2 \Rightarrow \frac{a}{b} < \frac{1}{\sqrt{2}} \Rightarrow \frac{BC}{AC} < \frac{1}{\sqrt{2}}.$$